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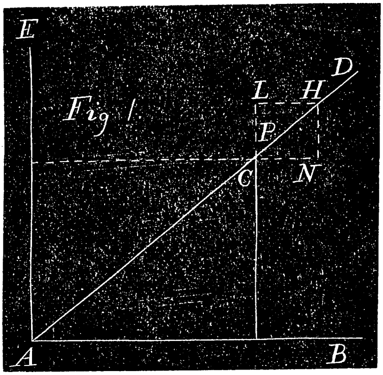
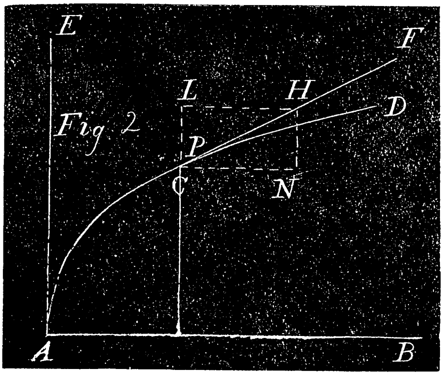
ON THE FIRST PRINCIPLES OF THE CALCULUS.

BY HIRAM COOK, C. E., NORWICH, CONN.

1. THE differential of a variable or function of a variable, is the *rate* at which such variable or function increases or decreases; that is, its *rate of variation*: and the object of the Differential Calculus is to determine the differential of a given function, in terms of the variable which enters such function, and the differential or rate of variation of that variable.

2. To illustrate the differential of a variable, that is, its rate of variation, we will suppose a particle P to be impelled from A toward D, along the line AD, and let the space described by P, at any instant, be represented by z; then the rate at which P is being impelled at the instant it coincides with any point in the line AD, as C, will represent the rate that z is increasing at the same instant, or dz the differential of z.





It will be observed that as P is being impelled from A toward D, it recedes from the line AB; therefore, if y represents the distance of P from the line AB at right angles thereto, the rate at which P is receding from AB at any instant, as when it coincides with the point C, will represent the rate that y is increasing at the same instant, or dy the differential of y. Also, if x represents the distance of P from the line AE, a perpendicular to AB, the rate at which P is receding from AE at any instant, as when it coincides with the point C, will represent the rate at which x is incaeasing at that instant, or dx the differential of x.

Now if P is left to itself at any point as C, that is, if the force or forces which impell it from A along the line AD, are wthdrawn at C, it is obvious that P will thereafter proceed uniformly at the rate it is being impelled at C, and, in fig. 1, still in the direction of D, but in fig. 2, in the direction of F, along the line CF, tangent to the curve at C. Therefore, if

P, under the supposition that it is left to itself at C, describes the space CH during the first instant of time after leaving the point C, CH will represent the value of dz, CL or NH, of dy, and CN or LH, of dx; for it is evident that if P proceeds uniformly from C toward D, fig. 1, or from C toward F, fig. 2, at the rate it was being impelled at C, it will also recede uniformly from the line AB at the same rate it was receding at C, and likewise from the line AE. Consequently if P describes CH during the first instant of its motion from C, it will recede from AB the distance CL or NH, and from AE the distance CN or LH in the same time. Hence, since

$$CH^2 = CN^2 + NH^2,$$

we have the following important equation; viz.,

$$dz^2 = dx^2 + dy^2.$$

3. Let the equation of the line CG be

$$y = ax + b \quad \dots\dots\dots\dots\dots \quad (1)$$

in which x represents any abscissa, as AD, y the corresponding ordinate DP, and b the distance AC. Then, drawing CE parallel to AD, we have

$$DE = AC = b,$$

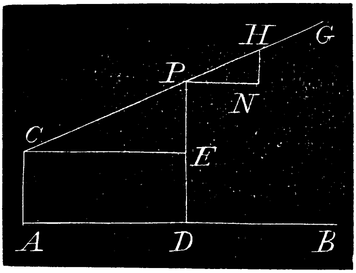
$$EP = DP - DE = y - b,$$

and

$$CE = AD = x.$$

Therefore, if we represent dx by PN and dy by NH, Art. 1, we shall have

$$x : y - b :: dx : dy;$$



whence

$$xdy = ydx - bdx,$$

or

$$dy = \frac{ydx - bdx}{x}.$$

Substituting for y, in the second member of the last equation, its value from (1), we have

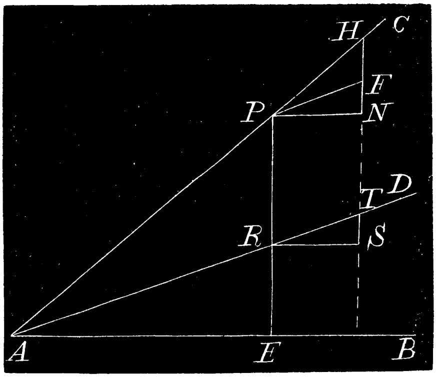
$$dy = \frac{axdx + bdx - bdx}{x},$$

or

$$dy = adx,$$

the differential of the primitve function, $y = ax + b$.

It will be seen that if $b = 0$, the differential will be the same. Hence:—*A constant quantity has no differential; that is, a constant term, connected with a variable by the sign plus or minus, will disappear in differentiating. Also, the differential of a variable multiplied by a constant quantity, is equal to the constant into the differential of the variable.*



4. If any ordinate of the line AC, as EP, is represented by u, and ER, that portion of EP which is the ordinate of the line AD, by y; then, as has been shown, Art. 1, du may be represented by NH, and dy by ST.

Now if PF is drawn parallel to RT, NF will be equal to ST, and consequently NH, the representation of the rate of increase of EP, will exceed ST, the representation of the rate of increase of ER, by FH: hence FH will represent the rate of increase of RP; that is, if RP is represented by z, dz will be represented by FH. Therefore, since

$$EP = u = ER + RP = y + z,$$

and $$NH = du = NF + FH = ST + FH = dy + dz,$$

we have $$du = dy + dz \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

for the differential of $$u = y + z.$$

If we transpose (2), we have $dy = du - dz$, which is the differential of

$$y = u - z,$$

as will be seen in the figure.

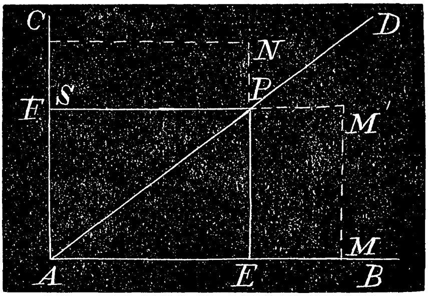
Hence:—*The differential of the sum or difference of two variables, is equal to the corresponding sum or difference of their differentials.*

5. Let AC (see Fig. on next page) be at right angles to AB, and suppose a point P to advance from A toward D, along the line AD, carrying with it the lines PR and PS, PR perpendicular to AB and PS perpendicular to AC; and suppose these lines so to increase in length, that R, the extremity of one, shall always be in the line AB, and S, the extremity of the other, shall always be in the line AC. Then, at any instant, as when R and S simultaneously arrive at the points E and F in the lines AB and AC, the line PR will have described the area APR, and the line PS the area APS.

Now it is evident that a right line, advancing in a direction at right angles to itself, will, at any instant, be describing an area equal to the length of the line at that instant, multiplied by the rate at which it is then advancing. Thus, if a line at the instant it is ten inches long, is advancing at the rate of two inches per second, it will then be describing an area at the rate of two times ten, or twenty square inches per second.

Hence, if RM or PM' represents the rate at which PR is advancing at the instant R arrives at E, the rectangle $RPM'M$ will represent the rate at which PR is then describing area, or the rate at which the area APR is then increasing. So also, if SN or PN' represents the rate at which PS is advancing at the instant S arrives at F, the rectangle $SNN'P$ will represent the rate at which the area APS is increasing at the same instant, viz., at the instant S arrives at F.

But if x represents the distance of R from A at any instant, and y the distance of S from A at the same instant; then, Art. 1, RM will be represented by dx and SP by dy. Therefere, since RP is equal to y and SP to x, ydx represents the rectangle $RP\,M'M$, and xdy the rectangle $SNN'P$; and consequently $xdy + ydx$ represents the rate at which the area of both APS and APR, or the area of the rectangle $ASPR$, is increasing at the instant R and S simultaneously arrive at the points E and F.



If we reprsent the area of the rectangle $ASPR$ by u, then, since the sides of the rectangle are represented by x and y, we have

$$u = xy. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (3)$$

But, since u represents the area, du the differential of u, must represent the rate of increase of the area, which has been shown to be $xdy + ydx$; therefore

$$du = xdy + ydx. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (4)$$

Hence:—*The differential of the product of two variables is equal to the first into the differential of the second, plus the second into the differential of the first.*

6. If in (3) we make y equal to x, u will be equal to x^2; also dy will be equrl to dx, for if y and x are always equal their rates of increase must necessarily be equal. Therefore, by substituting x for y and dx for dy in (4), we find the differential of

$$u = x^2$$

to be

$$du = xdx + xdx = 2xdx;$$

that is,

$$du = 2x^{2-1}dx.$$

If we make $y = x^2$, then will $dy = 2xdx$ and $u = x^3$. Therefore, by substituting x^2 for y and $2xdx$ for dy in (3), we find the differential of

$$u = x^3$$

to be

$$du = 2x^2dx + x^2dx = 3x^2dx;$$

or

$$du = 3x^{3-1}dx.$$

Again, if we make $y = x^3$, then will $dy = 3x^2dx$ and $u = x^4$. Therefore, by substituting in (3) x^3 for y, and $3x^2dx$ for dy, we find the differential of

$$u = x^4$$

to be $du = 3x^3dx + x^3dx = 4x^3dx$;

or $du = 4x^{4-1}dx.$

Proceeding in this manner, we shall find the differentials of

$u = x^5$ to be $du = 5x^{5-1}dx,$

$u = x^6$ " " $du = 6x^{6-1}dx,$

$u = x^7$ " " $du = 7x^{7-1}dx$;

from which it is evident that the differential of

$u = x^n$, in which n is any positive whole number, is $du = nx^{n-1}dx.$

Therefore the differential of $u = ax^n$ (by Art. 3) is

$$du = anx^{n-1}dx.$$

Hence:— *The differential of any positive integral power of a variable multiplied by a constant quantity, is equal to the constant into the product of the exponent denoting the power and the variable with its exponent diminished by unity, multiplied by the differential of the variable.*

Proceeding in like manner, the differentials of all algebraic functions may readily be determined.

7. Let us now consider the exponential function

$$u = a^x.$$

Assuming $a = 1 + c$ we have $u = (1 + c)^x$, or, developing $(1 + c)^x$ by the binomial theorem,

$$u = 1 + xc + \frac{x(x-1)}{2}c^2 + \frac{x(x-1)(x-2)}{2 \,.\, 3}c^3 + \&c. \quad . \; . \; . \; (5)$$

Differentiating this equation we obtain

$$du = \left(c + \frac{2x-1}{2}c^2 + \frac{3x^2 - 6x + 2}{2 \,.\, 3}c^3 + \&c.\right)dx:$$

Dividing each member of this equation by the corresponding member of (5), we have

$$\frac{du}{u} = \left(c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \&c.\right)dx.$$

Substituting for u its value a^x, and for c, its value $a - 1$, and multiplying by a^x, we find

$$du = a^x\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c.\right)dx;$$

but the series within the parenthesis, in this equation, is the Napierian logarithm of a, (see Brande, Art. Logarithm), therefore we have

$$du = a^x dx \log. a.$$

Hence:—*The differential of an exponential function, is equal to the Napierian logarithm of the constant of which the variable is the exponent, multiplied by the function itself into the differential of the exponent.*

8. Resuming the given function $u = a^x$ of the preceding article and taking the logarithm of both members, we have

$$\log u = x \log a, \qquad \text{whence}$$

$$x = \frac{\log u}{\log a}. \quad \ldots\ldots\ldots\ldots\ldots, \quad (6)$$

Now, Art. 7, the differential of $u = a^x$ is $du = a^x dx \log a$, whence

$$dx = \frac{du}{a^x \log a}, \text{ or, substituting } u \text{ for } a^x, \ dx = \frac{du}{u \log a}. \ \ldots\ldots \ (7)$$

Therefore, (7) is the differential of the logarithmic function (6).

If a is the base of a system of logarithms, then x is the logarithm of u in that system and $1 \div \log a$ is the modulus of the system. Therefore, if we represent $1 \div \log a$, in (6), by M, we have

$$dx = M\frac{du}{u}. \quad \ldots\ldots\ldots\ldots\ldots \ (8)$$

Hence:—*The differential of the logarithm of a variable is equal to the modulus of the system into the differential of the variable, divided by the variable.*

9. The modulus of the Napierian system is unity; therefore, if the logarithms are taken in that system (8) becomes

$$dx = \frac{du}{u};$$

that is, the differential of the Napierian logarithm of a variable, is the differential of the variable divide by the variable.

By the preceding method, all exponential and logarithmic functions may be differentiated.

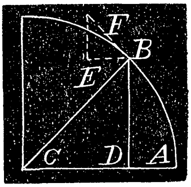
10. We will now consider the trigonometrical function $u = \sin x$, in which x is the arc and u its sine.

Let $AB = x$, $BD = u$, and $CB = R$; then, Art. 2, we have BF represented by dx and EF by du. Therefore, since the angle EBF is equal to the angle CBD and $CD = \cos x$, we have

$$R : \cos x :: dx : du,$$

or

$$du = \frac{\cos x \, dx}{R}.$$



Hence:—*The differential of the sine of an arc is equal to the cosine of the arc, into the differential of the arc, divided by the radius.*

In a similar manner we may find the following functions to have the respective differentials, viz.;

$u = \cos x$, $\quad du = -\dfrac{\sin x \, dx}{R}$; $\qquad u = \text{ver.sin}\, x$, $\quad du = \dfrac{\sin x \, dx}{R}$;

$u = \tan x$, $\quad du = \dfrac{R^2 \, dx}{\cos^2 x}$; $\qquad u = \cot x$, $\quad du = -\dfrac{R^2 \, dx}{\sin^2 x}$;

$u = \sec x$, $\quad du = \dfrac{\tan x \, dx}{\cos x}$; $\qquad u = \text{cosec}\, x$, $\quad du = -\dfrac{\cot x \, dx}{\sin x}$.

11. To determine the differential of the logarithmic sine of an arc, or of the function $u = \log\sin x$.
Differentiating, Arts. 9 and 10, we find

$$du = \frac{\cos x\,dx}{R\sin x}, \text{ or, since } \frac{\cos x}{R\sin x} = \frac{1}{\tan x},\ du = \frac{dx}{\tan x}.$$

In a similar manner we shall find the differentials of $u = \log\cos x$, and $u = \log\tan x$, to be, respectively,

$$du = -\frac{dx}{\cot x}, \quad \text{and } du = \frac{\tan x\,dx}{\sin^2 x}.$$

12. Finally we will take the function $z = \sin^{-1}x$, ... (8) in which z is the arc of which x is the sine. Hence, the equivalent relation will be $x = \sin z$; the differential of which, Art. 10, is

$$dx = \frac{\cos z\,dz}{R}; \text{ whence we have } dz = \frac{R\,dx}{\cos z}.$$

But since $\sin z = x$, and consequently $\cos z = \surd(R^2 - x^2)$, we have

$$dz = \frac{R\,dx}{\surd(R^2 - x^2)},$$

which is the differential of (8); i. e., the differential of the arc in functions of its sine. Similarly we shall find the differentials of $z = \cos^{-1}x$, $z = \tan^{-1}x$ &c., to be, respectively,

$$dz = -\frac{R\,dx}{\surd(R^2 - x^2)}, \quad dz = \frac{R^2dx}{R^2 + x^2}, \text{ \&c.}$$

Upon these principles the whole structure of the Differential Calculus may be erected so as to be readily comprehended by the ordinary student.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in No. five have been received as follows:

From Marcus Baker, 83, 84 & 85; E. S. Farrow, 84, 85, 86, 87 & 88; J. M. Greenwood and W. H. Baker, 83, 84, 86, 87 & 88; Henry Gunder, 84 & 86; Prof. W. W. Johnson, 86 & 91; Prof. H. T. J. Ludwick, 84, & 86; F. P. Matz, 86; Dr. A. B. Nelson, 84, 86, 87 & 88; O. D. Oathout, 84; Walter Siverly, 83, 84, 86, 87, 88, 89, 90 & 91; E. B. Seitz, 83 84 & 88; Prof. J. Scheffer, 83, 84, 85, 87 & 88; Prof. D. Trowbridge, 86; R. J. Adcock, 89.

83. "A point is given within two lines which form a given angle with one another. Required the shortest line which can be drawn through this point, terminated by the given lines."